Technical Comments

Comment on "A Practical Nondiverging Filter"

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Tarn and Zaborszky[1] introduce a modified form of the Kalman filter which holds considerable promise as a means of dealing with the divergence problem that often occurs in Kalman filter applications. As they point out, an exponential weighting of the noise covariance matrices has the effect of reducing the influence of old data in the estimation of the current state and thereby compensates for the cumulative effect of model errors. The computer simulations presented in Ref. 1 certainly indicate the effectiveness of this approach.

The purpose of this correspondence is to point out that the filter given does not correspond to the model given but rather to a model which exponentially weights Q_k and $P_{1/1}$ as well as R_k. Perhaps the easiest way to show this is to compare Tarn and Zaborszky's filter with the Kalman filter obtained for a model that includes the aged covariance matrices. This may be accomplished as follows. Let the current time n be fixed and define the covariance matrices as $Q_{k/n} = s^{n-k-1}Q_k$, $R_{k+1/n} = s^{n-k-1}R_k$ for $1 \leq k \leq n-1$ and $P_{1/n} = s^{n-1}P_{1/1}$. Except for these changes, all remaining details agree with Ref. 1. The Kalman filter equations for this model are

$$\hat{\mathbf{x}}_{k+1}{}^K = \Phi(t_{k+1},t_k)\hat{\mathbf{x}}_k{}^K + P_{k+1/k}{}^K H_k'(s^{n-k-1}R_{k+1} + H_kP_{k+1/k}{}^K H_k')^{-1}[y_{k+1} - H_k\Phi(t_{k+1},t_k)\hat{\mathbf{x}}_k{}^K]$$

$$P_{k+1/k+1}{}^K = P_{k+1/k}{}^K - P_{k+1/k}{}^K H_k'(s^{n-k-1}R_{k+1} + H_kP_{k+1/k}{}^K H_k')^{-1}H_kP_{k+1/k}{}^K$$

$$P_{k+1/k}{}^K = \Phi(t_{k+1},t_k)P_{k/k}{}^K\Phi'(t_{k+1},t_k) + s^{n-k-1}Q_k$$

$$P_{1/1}{}^K = s^{n-1}P_{1/1}$$

where a superscript K denotes the Kalman variables as distinguished from the variables $\hat{\mathbf{x}}_{n+1}$, $P_{n+1/n+1}$,$P_{n+1/n}$ appearing in the filter Eqs. (9–12) of Ref. 1.

It follows by an easy induction argument that

$$\begin{aligned} P_{k+1/k}{}^K &= s^{n-k-1}P_{k+1/k} \\ P_{k+1/k+1}{}^K &= s^{n-k-1}P_{k+1/k+1} \end{aligned} \tag{1}$$

But this implies that the gain

$$K_{k+1}{}^K = P_{k+1/k}{}^K H_k'(R_{k+1} + H_kP_{k+1/k}H_k')^{-1}$$

can be written as

$$K_{k+1}{}^K = P_{k+1/k}H_k'(R_{k+1} + H_kP_{k+1/k}H_k')^{-1}$$

for $1 \leq k \leq n$. But this is the same gain as used to obtain $\hat{\mathbf{x}}_k$, so it follows that $\hat{\mathbf{x}}_n{}^K = \hat{\mathbf{x}}_n$ as contended. The equality $P_n{}^K = P_n$ follows immediately from Eq. (1).

The proof given by Tarn and Zaborszky has errors in two places. In considering the unforced plant, they derive the correct recursive formulas for the covariance matrix $P_{n/n}$ but neglect to consider the initial covariance matrix $P_{1/1}$ that corresponds to this result. Further, when their proof is extended to include plant noise $\mathbf{u}_n$, the dependence of $P_{n+1/n+}$ on $Q_1, Q_2, \ldots, Q_{n-1}$ was not handled properly.

Fortunately, the model as modified here has a more meaningful interpretation than the model stated by Tarn and Zaborszky. Since the estimate $\hat{\mathbf{x}}_n$ depends on the dynamic model relating $\mathbf{x}_k$ to $\mathbf{x}_n$ as well as on the observation $\mathbf{z}_k$, it is appropriate to keep the weighting of $Q_{k/n}$ relative to $R_{k/n}$ fixed while at the same time decreasing the importance of the measurement $\mathbf{z}_k$ in the determination of $\hat{\mathbf{x}}_n$.

In a forthcoming paper,[2] the filter is derived for a more general weighting coefficient. Also, the filter equations for a continuous system are presented, and asymptotic properties and bounds for the error covariance matrix are given. In addition, an adaptive procedure which can be used for the selection of values for the weighting factor is discussed that is based on the behavior of the measurement residual.

Finally, it should be noted that the idea of exponentially aging data in a recursive filter was discussed by Fagin[3] for the discrete case with no plant noise (i.e., $Q_k \equiv 0$ for all k).

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[1] Tarn, T. J. and Zaborszky, J., "A Practical, Nondiverging Filter," *AIAA Journal*, Vol. 8, No. 6, June 1970, pp. 1127–1133.

[2] Sorenson, H. W. and Sacks, J. E., "Recursive Fading Memory Filtering," to be published in *Information Sciences*.

[3] Fagin, S. F., "Recursive Linear Regression Theory, Optimal Filter Theory, and Error Analysis of Optimal Systems," *1964 IEEE Convention Record*, pp. 216–240.

Reply by Authors to J. E. Sacks and H. E. Sorenson

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The approach which is adopted in Ref. 1 is a natural one. The interpretation stated in Ref. 2, of course, could be considered as another interpretation, but it has the difficulty of justifying why to choose the same s factor for both plant noise and measurement noise. Besides, as shown in Ref. 2, using this alternative interpretation does not affect the final result. Further, from the recursive formula for the covariance matrix $P_{n/n}$ [Eqs. (10) and (11) in Ref. 1], the fading effect to the initial covariance matrix $P_{1/1}$ is automatically considered, which seems to answer the comment regarding $P_{1/1}$. Also, it would seem that the dependence of $P_{n/n}$ on $Q_1, \ldots, Q_{n-1}$ was handled properly. Since the plant noise is an independent random sequence, it is clear from the state equation that the only effect the plant noises can have is to increase the extrapolated covariance matrix.

Really, the data discounting procedure has a long history; it has long been used in Fourier analysis. Fagin[3] discussed the exponential aging of data for the static case ($\Phi = I$) and with no plant noise. The extension to the dynamic case could be considered important.

As stated in Ref. 1, the factor s could be selected empirically. Sorenson and Sacks in a forthcoming paper[4] indicate one interesting way to select the fading factor for a scalar system without plant noise. As stated in Ref. 4, "of course,

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it is reasonable to expect that multidimensional systems will require more complex procedures for choosing the fading factor if for no other reason than the problems of observability and controllability that are attendant to such systems." We agree that choosing the fading factor adaptively is likely to prove a very complex problem. This seems to contradict the effectiveness and simplicity requirement of the filter.

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[1] Tarn, T. J. and Zaborszky, J., "A Practical, Nondiverging Filter," *AIAA Journal*, Vol. 8, No. 6, June 1970, pp. 1127–1133.

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[3] Fagin, S. F., "Recursive Linear Regression Theory, Optimal Filter Theory and Error Analysis of Optimal Systems," *IEEE International Convention Record*, March 1964, pp. 216–240.

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Comment on "Choked Flow: A Generalization of the Concept and Some Experimental Data"

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IN Ref. 1 there is derived the chocking condition for two or more streams flowing through a De Laval type duct. This condition is arrived at from the physical requirement that for a given total mass flow rate the total equivalent momentum flux is a minimum; or alternatively, for a given total equivalent momentum flux the total mass flow rate is a maximum.

The same condition is derived here without having to resort to physical requirements, that is, on a purely mathematical basis. The choking condition is derived from the requirements at the singular points of the conservation equations. The assumptions made here are the same as Ref. 1, specifically, that the flow is steady, one dimensional and the gases are assumed ideal.

Following the development of Ref. 2, the conservation equations may be written as follows. The Mach number variation of each stream is given by

$$dM_i = -\frac{\{1 + [(\gamma_i - 1)/2]M_i^2\}}{1 - M_i^2}\frac{M_i}{A_i}dA_i + \sum_{j=1}^{N}\frac{f_j(\gamma_i,M_i)}{1 - M_i^2}\frac{M_i}{2}\frac{dZ_j}{Z_j} \qquad i = 1,2,\ldots,n$$

where the functions $f_j(\gamma_i,M_i)$ are listed in Table 8.1 of Ref. 2 and the dZ_j represent the forcing terms such as heat addition, mass injection, friction etc.... Since the pressure of all the streams is the same, i.e., $p_i = p$, the area variation of each stream may be calculated from

$$dA_i = \frac{1 - M_i^2}{\gamma_i M_i^2}\frac{A_i}{p}dp - \frac{A_i}{\gamma_i M_i^2}\sum_{j=1}^{N} g_j(\gamma_i,M_i) \times \frac{dZ_j}{Z_j} \qquad i = 1,2,\ldots,n$$

where the $g_j(\gamma_i,M_i)$ are also listed in Table 8.1 of Ref. 2. The area constraint relation

$$\sum_{i=1}^{n} A_i = A \text{ or } \sum_{i=1}^{n} dA_i = dA$$

supplies the final equation of a system of $2n + 1$ equations for the $2n + 1$ unknowns dM_i, dA_i, dp.

The singular point of this system occurs when the determinant of the homogeneous system vanishes. It is evident from the preceding that the determinant is equal to

$$D = \begin{vmatrix} \frac{\gamma M^2}{1 - M_i^2}\frac{p}{A_i} & & 0 & 1 \\ & \ddots & & \vdots \\ 0 & & \frac{\gamma_n M_n^2}{1 - M_n^2}\frac{p}{A_n} & 1 \\ 1 & \cdots & 1 & 0 \end{vmatrix}$$

After expansion by the nth column and then the resulting cofactors by their respective r-th column, this can be shown to be equal to

$$D = \sum_{i=1}^{n}\left(\prod_{\substack{i=1 \\ i\neq r}}^{n}\frac{\gamma_i M_i^2}{1 - M_i^2}\frac{p}{A_i}\right) = \left(\prod_{i=1}^{n}\frac{\gamma_i M_i^2}{1 - M_i^2}\frac{p}{A_i}\right) \times \left(\sum_{r=1}^{n}\frac{1 - M_r^2}{\gamma_r M_r^2}\frac{A_r}{p}\right)$$

The requirement that the determinant vanishes gives the choking condition as

$$\sum_{i=1}^{n}\frac{1 - M_i^2}{\gamma_i M_i^2}A_i = 0$$

This is seen to be the same result as obtained in Ref. 1. Given this condition and with the mass flux fixed, it can be shown that the momentum flux

$$J = \sum_{i=1}^{n}(pA_i + \dot{m}_i u_i)$$

is a minimum, i.e., $dJ = 0$ when $dZ_j = 0$.

References

[1] Hoge, H. J. and Segars, R. A., "Choked Flow: A Generalization of the Concept and Some Experimental Data," *AIAA Journal*, Vol. 3, No. 12, Dec. 1965, pp. 2177–2183.

[2] Shapiro, A. H., *The Dynamics and Thermodynamics of Compressible Fluid Flow*, Ronald Press, New York, Vol. 1, 1953, p. 228.

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Reply by Authors to A. M. Agnone

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IT is interesting to see that the conditions of generalized choking can be derived in different ways. We thank Mr. Agnone for his straight-forward mathematical treatment. After our paper appeared, Bernstein, Heiser, and Hevenor[1] also published a paper on generalized choking. In some respects their method of attack is intermediate between Mr. Agnone's and ours.

Reference

[1] Bernstein, A., Heiser, W. H., and Hevenor, C., "Compound-Compressible Nozzle Flow," *Transactions of the ASME E34, Journal of Applied Mechanics*, Sept. 1967, pp. 548–54.

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